Talk on my Diploma Thesis:

“Extreme egalitarian weights on the residue class ring of integers modulo m”

original, german: „Extremale egalitäre Gewichte auf $\mathbb{Z}_m$“

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Contents

1 Definitions

In this part the basic definitions are given.

1.1 Weight Functions

Let $\mathbb{Z}_m$ be the (commutate) residue class ring of integers modulo m with $2 \leq m \in \mathbb{N}$.

Definition 1.1 (weight) *$w : \mathbb{Z}_m \to \mathbb{R}$ is called weight function*[1] *or simply weight*[2]*, iff:*

(W I) $w(x) = 0 \iff x = 0$

(W II) $\forall x \in \mathbb{Z}_m : w(-x) = w(x)$

(W III) $\forall x, y \in \mathbb{Z}_m : w(x + y) \leq w(x) + w(y)$

Remark 1.1 *It follows as a further predicate:*

(1) $\forall x \in \mathbb{Z}_m : w(x) \geq 0$

Definition 1.2 (egalitarian weight) *$w : \mathbb{Z}_m \to \mathbb{R}$ is called egalitarian weight*[3]*, iff:*

(EW I) *w is weight (def. 1.1)*

(EW II) $\exists \zeta \in \mathbb{R} : \forall U \lhd \mathbb{Z}_m$ *with* $U \neq \{0\}$:[4]

$$\sum_{x \in U} w(x) = \zeta |U|$$

Definition 1.3 (normalized egalitarian weight) *An egalitarian weight with $\zeta = 1$ is called normalized*[5].

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[1]german: `Gewichtsfunktion`

[2]german: `Gewicht`

[3]german: `egalitäres Gewicht`

[4]$U \lhd \mathbb{Z}_m$ means, that U is subgroup of $\mathbb{Z}_m$. Every subgroup is possibly represented by a divider $t|m$ as: $U = \{x \in \mathbb{Z}_m : x = k \cdot t\}$

[5]german: `normiertes egalitäres Gewicht`

Definition 1.4 (strongly egalitarian weight) *$w : \mathbb{Z}_m \to \mathbb{R}$ is called strongly egalitarian weight*[6]*, iff:*

(SEW I) *w is weight (def. 1.1)*

(SEW II) $\exists \zeta \in \mathbb{R} : \forall U \lhd \mathbb{Z}_m$ *with* $U \neq \{0\}$:

$$\forall \text{ coset } z + U \text{ with } z \in \mathbb{Z}_m :$$

$$\sum_{x \in z+U} w(x) = \zeta |U|$$

Remark 1.2 *Every strongly egalitarian weight is also an egalitarian weight.*

Definition 1.5 (normalized strongly egalitarian weight) *A strongly egalitarian weight with $\zeta = 1$ is called normalized*[7]*.*

Remark 1.3 *$w : \mathbb{Z}_m \to \mathbb{R}$ is normalized (strongly) egalitarian weight. $\Longrightarrow$ $\overline{w} : \mathbb{Z}_m \to \mathbb{R}, x \mapsto \zeta \cdot w(x)$ is (strongly) egalitarian.*

Remark 1.4 *Contradiction for $m = 6k$:*
for $m \equiv 0 \, (mod\ 6)$: $\{0; \frac{m}{6}; \frac{2m}{6}; \frac{3m}{6}; \frac{4m}{6}; \frac{5m}{6}\} \lhd \mathbb{Z}_m$, so:

$$\underbrace{w(0)}_{=0} + w\left(\frac{m}{6}\right) + \underbrace{w\left(\frac{m}{3}\right)}_{=\frac{3}{2}} + \underbrace{w\left(\frac{m}{2}\right)}_{=2} + \underbrace{w\left(\frac{m}{3}\right)}_{=\frac{3}{2}} + w\left(\frac{m}{6}\right) = 6$$

$$\Longrightarrow w\left(\frac{m}{6}\right) = \frac{1}{2}$$

The triangle inequality 1.1 (W III) says:

$$\frac{3}{2} = w\left(\frac{m}{3}\right) = w\left(\frac{m}{6} + \frac{m}{6}\right) \leq 2w\left(\frac{m}{6}\right) = 1$$

Therefore we have a contradiction.

For $m \equiv 0 \, (mod\ 6)$ there exists no egalitarian and no strongly egalitarian weight.

[6] german: `vollständig egalitäres Gewicht`
[7] german: `normiertes vollständig egalitäres Gewicht`

1.2 Weight Vectors

Every weight function (def. 1.1) can be interpreted as a vector: [8]

$$x = (x_0, x_1, \ldots, x_{m-1})^T \in \mathbb{R}^m$$

$$\forall i \in [0; m-1] \cap \mathbb{Z} : [i]_m \in \mathbb{Z}_m \wedge w([i]_m) = x_i$$

In the same way as for a weight function the following words are defined:

- weight vector
- egalitarian weight vector
- strongly egalitarian weight vector

With these definitions a weight is nothing else than a dedicated point in the $\mathbb{R}^m$. All restrictions to these points are linear; therefore the set of all (strongly) egalitarian weight vectors for one m moulds a polytope.

[8] $\mathbb{Z}_m$ can be interpreted as $[0; m-1] \cap \mathbb{Z}$.

2 Examples

2.1 Weight for $m = 5$

- $x \in \mathbb{R}^5$
- $I = \{1; 2\}$

(G1) $\begin{cases} x_0 = 0 \\ x_1 > 0 \\ x_2 > 0 \end{cases}$

(G2) $\begin{cases} x_2 \leq x_1 + x_1 \\ x_1 \leq x_2 + x_2 \end{cases}$

(G3)

(E2) $x_1 + x_2 + x_2 + x_1 = 5$

(VE2) $x_1 + x_2 + x_2 + x_1 = 5$

Remember: $x_0 = 0$, $x_4 = x_1$ and $x_3 = x_2$
Every normalized egalitarian weight is normalized strongly egalitarian, too. The set of all normalized egalitarian weights on $\mathbb{Z}_5$ has one degree of freedom - see fig. 1.

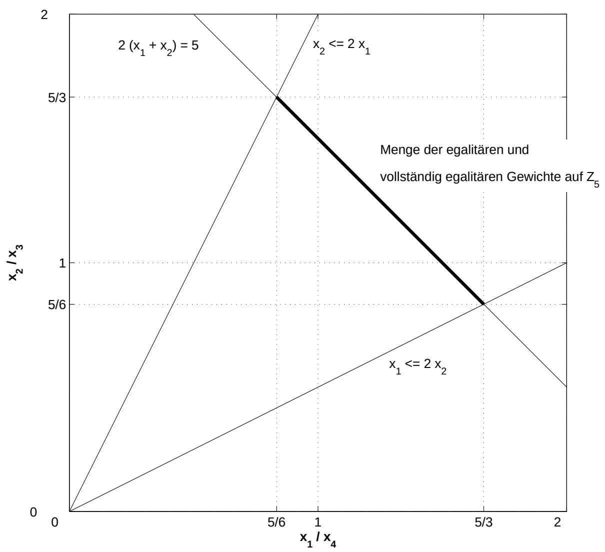


Figure 1: normalized egalitarian and strongly egalitarian weights on $\mathbb{Z}_5$ as a line in $\mathbb{R}^2$

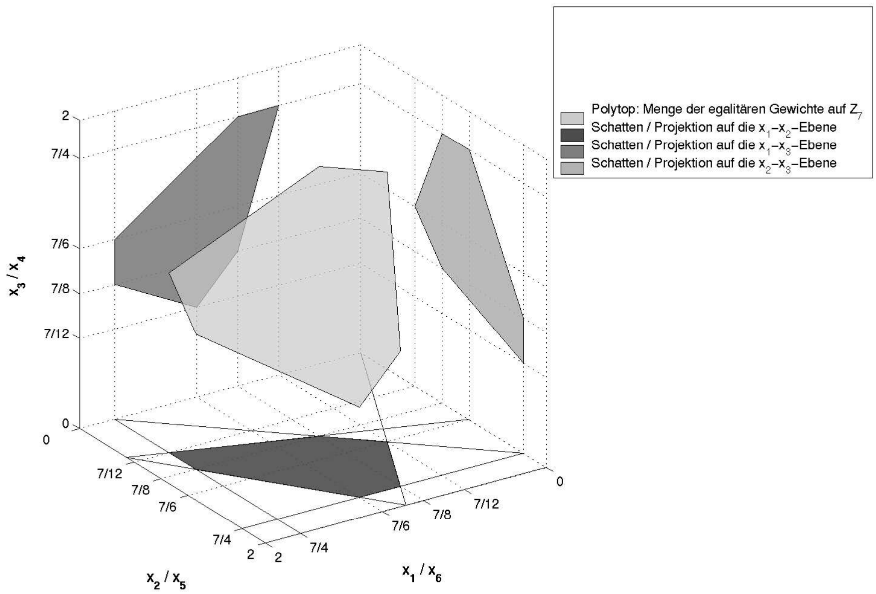


Figure 2: normalized egalitarian weights on $\mathbb{Z}_7$ as a surface in $\mathbb{R}^3$

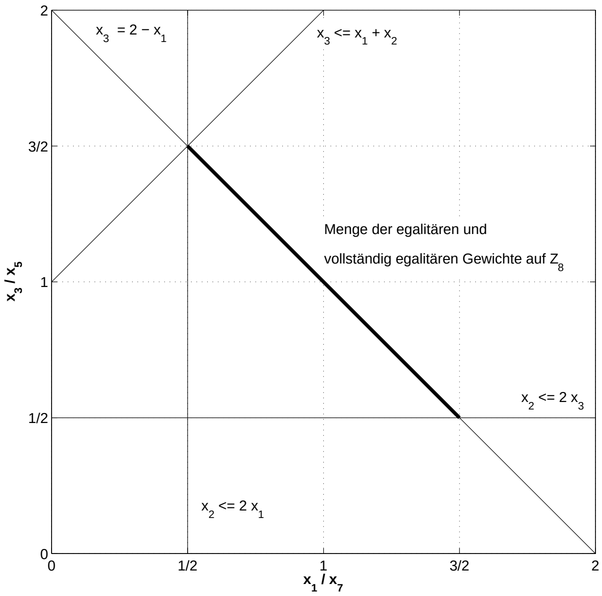


Figure 3: normalized egalitarian and strongly egalitarian weights on $\mathbb{Z}_8$ as a line in $\mathbb{R}^2$

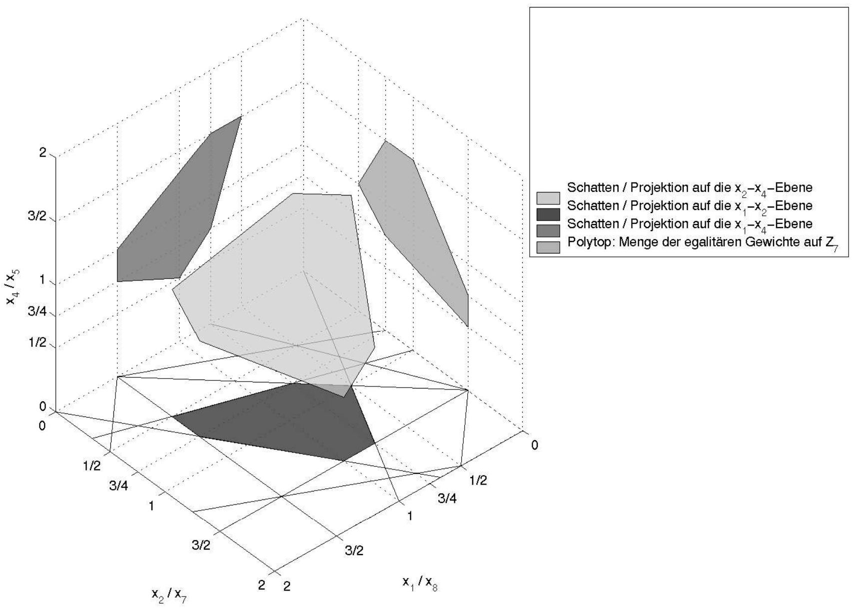


Figure 4: normalized egalitarian and strongly egalitarian weights on $\mathbb{Z}_9$ as a surface in $\mathbb{R}^3$

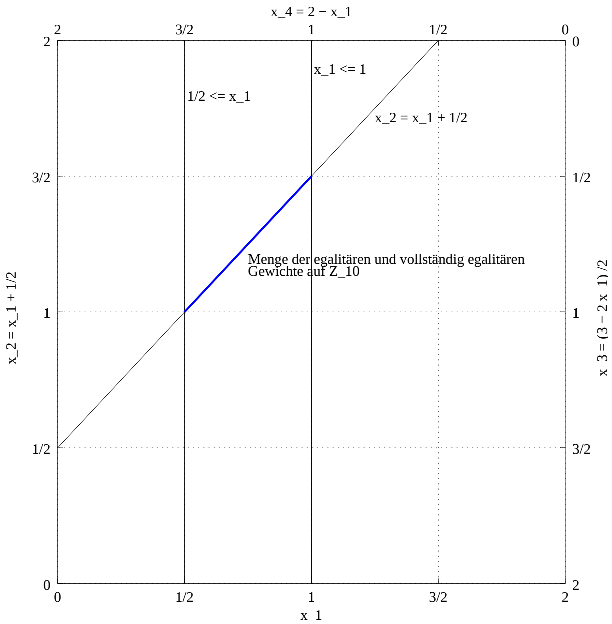


Figure 5: normalized egalitarian and strongly egalitarian weight on $\mathbb{Z}_{10}$

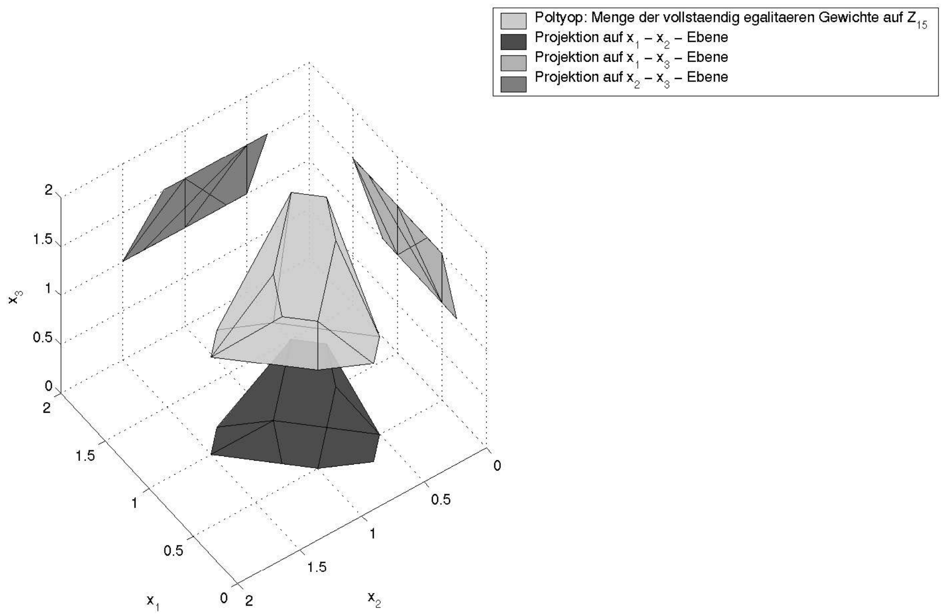


Figure 6: normalized strongly egalitarian weights on $\mathbb{Z}_{15}$

3 Equations - Inequations

If you look at the numbers of equations and inequations of the problems, you can easily see:

(W I) $w(x) = 0 \iff x = 0$

is not relevant because it is known.

(W II) $\forall x \in \mathbb{Z}_m : w(-x) = w(x)$

are $\boxed{m \text{ equations}}$ and

(W III) $\forall x, y \in \mathbb{Z}_m : w(x + y) \leq w(x) + w(y)$

are $\boxed{m^2 \text{ inequations}}$.

(EW I) w is weight (def. 1.1)

(EW II) $\exists \zeta \in \mathbb{R} : \forall U \lhd \mathbb{Z}_m$ with $U \neq \{0\}$:

$$\sum_{x \in U} w(x) = \zeta |U|$$

are $\boxed{|T| \text{ equations}}$ with $T := \{d \in \mathbb{N} : d|m \wedge d > 1\}$.

(SEW I) w is weight (def. 1.1)

(SEW II) $\exists \zeta \in \mathbb{R} : \forall U \lhd \mathbb{Z}_m$ with $U \neq \{0\}$:

$\forall$ coset $z + U$ with $z \in \mathbb{Z}_m$:

$$\sum_{x \in z+U} w(x) = \zeta |U|$$

are $\boxed{|T| \cdot (m-1) \text{ equations}}$ with the same T as above.
But of course there is a lot of redundancy - for example see the inequations for a weight in fig. 7 or the equations for strongly egalitarian in fig. 8 and fig. 9.

Figure 7: Redundancy in the inequations in def. 1.1 (W III)

$x+y$ x / y	0	1	2	3	4	$\cdots$	4	3	2	1
0	0	1	2	3	4	$\cdots$	4	3	2	1
1	1	**2**	**3**	**4**	**5**	$\cdots$	**3**	**2**	1	0
2	2	3	**4**	**5**	**6**	$\cdots$	2	**1**	0	1
3	3	4	5	**6**	**7**	$\cdots$	**1**	0	1	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-4=4$	4	3	2	1	0	$\cdots$	8	7	6	5
$m-3=3$	3	2	1	0	1	$\cdots$	7	6	5	4
$m-2=2$	2	1	0	1	2	$\cdots$	6	5	4	3
$m-1=1$	1	0	1	2	3	$\cdots$	5	4	3	2

In the first column are written the possible values of $y \in \mathbb{Z}_m$ — $m-1$ is here equal to 1, because $w(y) = w(-y)$. In the first line is the same for $x \in \mathbb{Z}_m$. In the points of intersection you see the $x+y$ in the meaning of $w(x+y)$. Therefore again $m-1$ can represented as 1.
You can see 2 kinds of symmetries: From the upper left corner to the down right corner you have the symmetry axis due to the commutativity of $w(x)+w(y) = w(y)+w(x)$. From down left to upper right you have the symmetry axis due to $w(x) = w(-x)$.
The bold numbers mark the inequations for a weight without any obvious redundancy.

Therefore it is helpful to reduce these numbers - especially the number of inequations. It is possible to get:

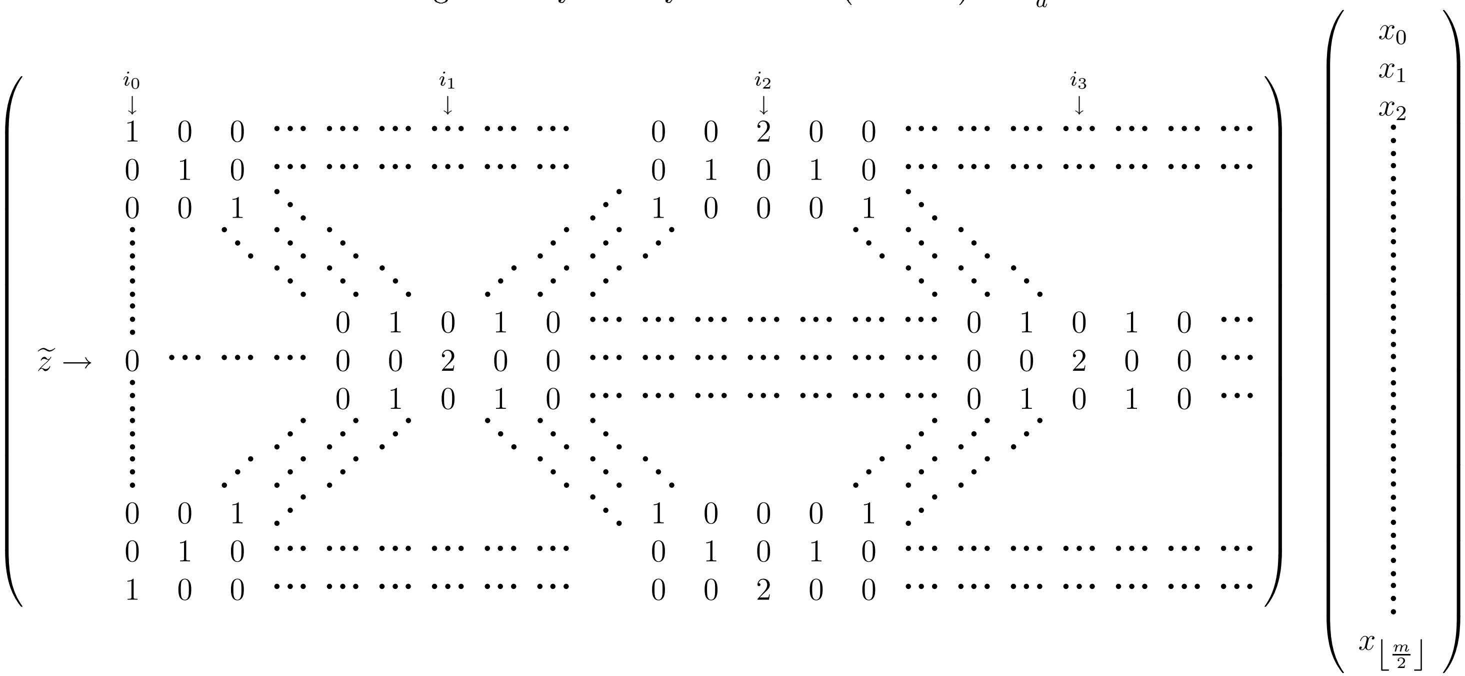
$d_1 = \left\lfloor \frac{m}{2} \right\rfloor$ number of unknowns, due to (W II)

$d_2 = \frac{m^2-4m}{4} + \begin{cases} 1 & \text{with } m \equiv 0 \pmod 2 \\ \frac{3}{4} & \text{with } m \equiv 1 \pmod 2 \end{cases}$ number of inequations for a weight

$d_3 = |T|$ number of equations for egalitarian

$d_4 = \sum_{d \in T_{\mathbb{P}}} \left\lfloor \frac{m}{2d} \right\rfloor + |T_{\mathbb{P}}|$ number of equations for strongly egalitarian

Figure 8: Symmetry in def. 1.4 (SEW II) for $\frac{m}{d}$ even:



$$\text{with} \quad \widetilde{z} = \frac{m}{2d} + 1 \quad \text{and} \quad \begin{aligned} i_0 &= 1 \\ i_1 &= i = \frac{m}{2d} \\ i_2 &= i = \frac{m}{d} \\ i_3 &= i = \frac{3m}{2d} \end{aligned}$$

Figure 9: Symmetry in def. 1.4 (SEW II) for $\frac{m}{d}$ odd:

$$
\begin{array}{c}
\\ \\ \\ \\ \\ \\ \widetilde{z} \rightarrow \\ \\ \\ \\ \\ \\
\end{array}
\left(
\begin{array}{cccccccccccccccccc}
\overset{i_0}{\downarrow} & & & & \overset{i_1}{\downarrow} & & & & & & \overset{i_2}{\downarrow} & & & & \overset{i_3}{\downarrow} & & & \\
1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & & 0 & 0 & 2 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & & 0 & 1 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 1 & \ddots & & & & \iddots & 1 & 0 & 0 & 0 & 1 & \ddots & & & & \\
\vdots & & \ddots & \ddots & \ddots & & \iddots & \iddots & & & & & & \ddots & \ddots & & & \\
\vdots & & & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & 0 & 1 \; 0 \; \cdots \\
0 & \cdots & \cdots & 0 & 0 & 1 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 1 & 0 \; 0 \; \cdots \\
\vdots & & & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & 0 & 1 \; 0 \; \cdots \\
\vdots & & \iddots & \iddots & \iddots & & \ddots & \ddots & & & & & & \iddots & \iddots & & & \\
0 & 0 & 1 & \iddots & & & & \ddots & 1 & 0 & 0 & 0 & 1 & \iddots & & & & \\
0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & & 0 & 1 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & & 0 & 0 & 2 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\right)
\begin{pmatrix}
x_0 \\ x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{\lfloor \frac{m}{2} \rfloor}
\end{pmatrix}
$$

with $\quad \widetilde{z} = \left\lfloor \frac{m}{2d} + 1 \right\rfloor \quad$ and

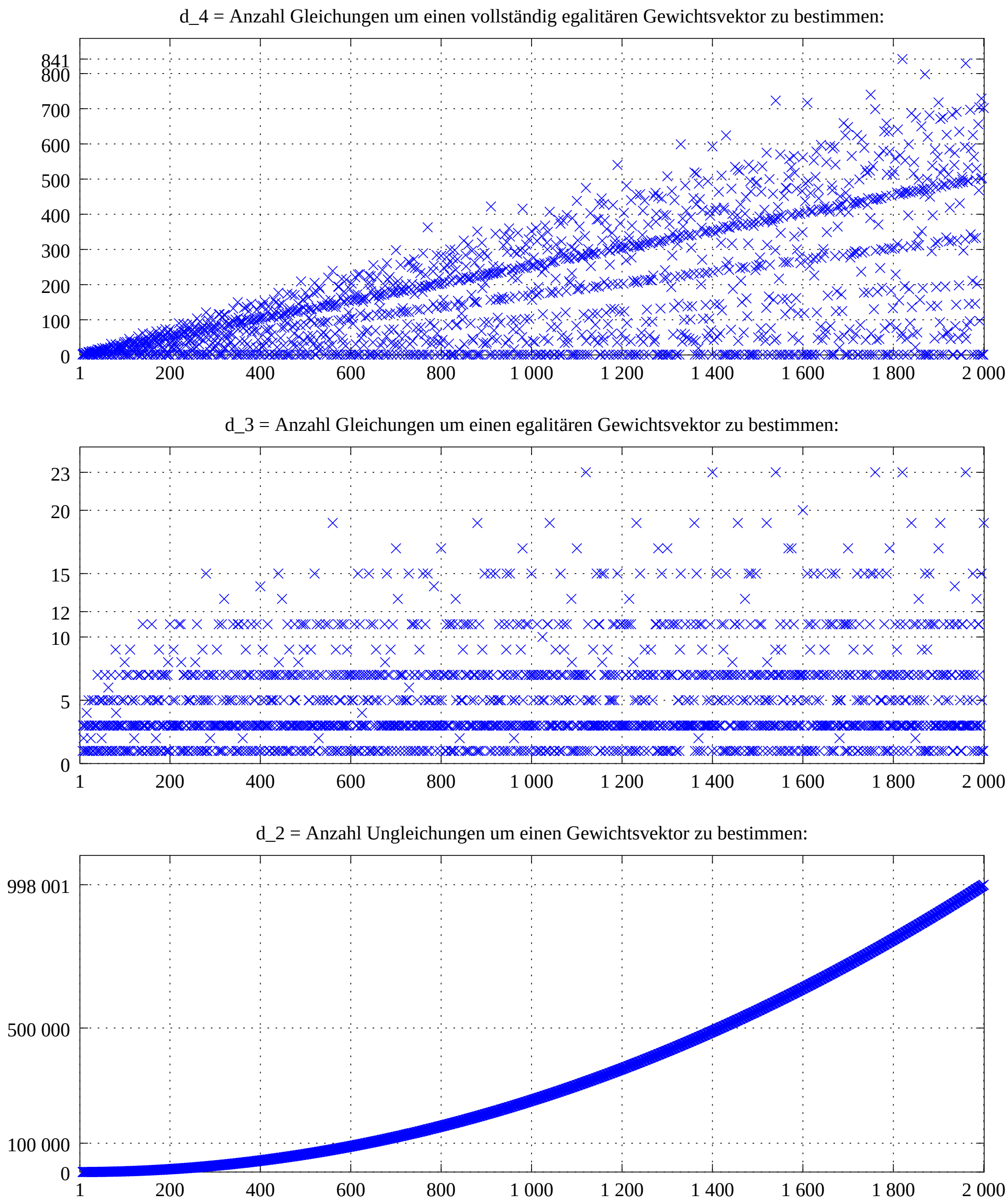
$$
\begin{aligned}
i_0 &= 1 \\
i_1 &= i = \left\lfloor \frac{m}{2d} \right\rfloor \\
i_2 &= i = \frac{m}{d} \\
i_3 &= i = \left\lfloor \frac{3m}{2d} \right\rfloor
\end{aligned}
$$

$$\text{with}: \quad T := \{d \in \mathbb{N} : d|m \,\wedge\, d > 1\}$$

$$T_{\mathbb{P}} := \{d \in \mathbb{P} : d|m\}$$

To find weights for $m \geq 500$ you still have more than 62001 inequations.

Figure 10: Number of equations and inequations for weights



4 Extrema

4.1 Definitions for Extremum

Definition 4.1 (normalisator map, extrema) *The function $\zeta(w)$ maps the egalitarian (resp. strongly egalitarian) weight to this ζ in the definition (def. 1.2, 1.4).*

$$\zeta(w) := \frac{1}{m} w(\mathbb{Z}_m) := \frac{1}{m} \sum_{x \in \mathbb{Z}_m} w(x), \text{ with } w \text{ (strongly) egalitarian}$$

Therewith you can write down for every weight w its normalized weight $\frac{w}{\zeta(w)}$.

$$\zeta_{min}(w) := \min \left\{ \frac{w(x)}{\zeta(w)} : w(x) \neq 0 \right\}$$

$$\zeta_{max}(w) := \max \left\{ \frac{w(x)}{\zeta(w)} : w(x) \neq 0 \right\}$$

Definition 4.2 (shortcuts for extrema)

$$_{min}\zeta_{egalitarian} := min\{\zeta_{min}(w) : w \text{ is egalitarian weight}\}$$

$$^{max}\zeta_{egalitarian} := max\{\zeta_{max}(w) : w \text{ is egalitarian weight}\}$$

$$_{min}\zeta^{strongly}_{egalitarian} := min\{\zeta_{min}(w) : w \text{ is strongly egalitarian weight}\}$$

$$^{max}\zeta^{strongly}_{egalitarian} := max\{\zeta_{max}(w) : w \text{ is strongly egalitarian weight}\}$$

4.2 Optimization

To get the maximum or minimum as to x_1 from a normalized egalitarian and strongly egalitarian weight respectivly you can solve the linear program:

$$c^T \cdot x \longrightarrow \text{max. / min.}$$

$$U \cdot x \leq 0$$

$$G \cdot x = b$$

$$x \geq 0; \qquad c^T = (1, 0, \ldots, 0)$$

There is U from the inequations in theorem 5.1 (G2), G(3) and the equations G, b from theorem 5.2 and 5.3 respectivly. c is the first column from the identity matrix. To find the extremum in another direction, say i, you choose the i^{th} column.

To get exact results without rounding errors it is necessary to calculate with rational numbers.[9]

Consequently it is impossible to get a great deal of results in a simply way. The answer is to guess an amount of suitable inequations at the beginning and then to add recursively the violated ones.[10]

4.3 Program, Results

In my diploma thesis I wrote a program in C and used integers with unlimited precision from the library CLN (Class Library for Numbers[11]). To calculate with these classes I had to implement the simplex algorithm and helper applications, like canceling down rational numbers[12] by the Euclidean algorithm or to find all (prime) dividers of an integer — see fig. 11.

The running time of the program is like the number of inequations for a weight — see again fig. 10.

In the tables 1 and 2 and in fig. 12 you can see some results of the numerical experiments.

In general only a few facts are known — see table. 3.

With the numerical experiments it was possible to make the hypothesis in fig. 13.

[9] With machine accuracy I had rounding errors already for $m = 11$. In the calculation of all extreme egalitarian weights up to 160 the biggest number in extensions needed about 80 bits of memory.

[10] I calculated all extreme weights up to 194 and all minima up to 357. On the whole there are up to 1155 results.

[11] http://www.ginac.de/CLN

[12] CLN has directly implemented rational numbers, but for memory efficiency it reduced the numerator and the denominator in every step. This was to slow for my project.

Figure 11: Canceling down with the Euclidean algorithm

```
void kuerze(cl_I *a, cl_I *b);
/* -------------------------------------------------
Es wird a und b gekuerzt.
Parameter:
  cl_I *a    : ganze Zahl a als Zaehler
  CL_I *b    : ganze Zahl b als Nenenr
0/b liefert 0/1;
a/0 fuehrt zum Programmabbruch;
das Vorzeichen wird in a gespeichert, d. h. hinterher: b > 0
-----------------------------------------------------
Autor: Daniel Mohr
----------------------------------------------------- */
void kuerze(cl_I *a, cl_I *b)
{
  cl_I zaehler, nenner, tmp;
  assert((*b) != 0);
  zaehler = abs((*a));
  nenner = abs((*b));
  if ((*a) != 0)
    {
      if ((zaehler != 1) && (nenner != 1))
        {
          /* GGT von zaehler und nenner wird mittels
             Euklid'schen Algorithmus bestimmt: */
          while(nenner != 0)
            {
              tmp = nenner;
              /* nenner = (cl_I) zaehler % nenner; */
              nenner = zaehler - floor1(zaehler,nenner) * nenner;
              zaehler = tmp;
            }
          /* fuer die Hilfsvariable gilt: zaehler = ggT(a,b) */
          /* nun kann gekuerzt werden: */
          (*a) = exquo((*a),zaehler);
          (*b) = exquo((*b),zaehler);
        }
      if ((*b)<0)
        {
          (*a) = -(*a);
          (*b) = -(*b);
        }
    } else {
      (*a) = 0L;
      (*b) = 1L;
    }
}
```

m	egalitarian ζ_{min}	egalitarian ζ_{max}	strongly egalitarian ζ_{min}	strongly egalitarian ζ_{max}
4	1	2	1	2
5	$\frac{5}{6}$	$\frac{5}{3}$	$\frac{5}{6}$	$\frac{5}{3}$
7	$\frac{7}{12}$	$\frac{7}{4}$	$\frac{7}{12}$	$\frac{7}{4}$
8	$\frac{1}{2}$	2	$\frac{1}{2}$	2
9	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
10	$\frac{1}{2}$	2	$\frac{1}{2}$	2
11	$\frac{11}{30}$	$\frac{11}{6}$	$\frac{11}{30}$	$\frac{11}{6}$
13	$\frac{13}{42}$	$\frac{13}{7}$	$\frac{13}{42}$	$\frac{13}{7}$
14	$\frac{1}{3}$	2	$\frac{1}{3}$	2
15	$\frac{3}{10}$	$\frac{5}{3}$	$\frac{1}{3}$	$\frac{3}{2}$
16	$\frac{1}{4}$	2	$\frac{1}{4}$	2
17	$\frac{17}{72}$	$\frac{17}{9}$	$\frac{17}{72}$	$\frac{17}{9}$
19	$\frac{19}{90}$	$\frac{19}{10}$	$\frac{19}{90}$	$\frac{19}{10}$
20	$\frac{1}{4}$	2	$\frac{1}{4}$	2
21	$\frac{3}{14}$	$\frac{38}{21}$	$\frac{1}{4}$	$\frac{3}{2}$
22	$\frac{1}{5}$	2	$\frac{1}{5}$	2
23	$\frac{23}{132}$	$\frac{23}{12}$	$\frac{23}{132}$	$\frac{23}{12}$
25	$\frac{1}{6}$	$\frac{25}{13}$	$\frac{1}{6}$	$\frac{25}{13}$
26	$\frac{1}{6}$	2	$\frac{1}{6}$	2
27	$\frac{1}{6}$	$\frac{36}{19}$	$\frac{1}{6}$	$\frac{3}{2}$
28	$\frac{1}{6}$	2	$\frac{1}{6}$	2
29	$\frac{29}{210}$	$\frac{29}{15}$	$\frac{29}{210}$	$\frac{29}{15}$
31	$\frac{31}{240}$	$\frac{31}{16}$	$\frac{31}{240}$	$\frac{31}{16}$
32	$\frac{1}{8}$	2	$\frac{1}{8}$	2
33	$\frac{3}{22}$	$\frac{36}{19}$	$\frac{1}{7}$	$\frac{3}{2}$
34	$\frac{1}{8}$	2	$\frac{1}{8}$	2
35	$\frac{5}{42}$	$\frac{23}{12}$	$\frac{1}{8}$	$\frac{19}{10}$
37	$\frac{37}{342}$	$\frac{37}{19}$	$\frac{37}{342}$	$\frac{37}{19}$
38	$\frac{1}{9}$	2	$\frac{1}{9}$	2
39	$\frac{3}{26}$	$\frac{86}{45}$	$\frac{1}{8}$	$\frac{3}{2}$
40	$\frac{1}{8}$	2	$\frac{1}{8}$	2
41	$\frac{41}{420}$	$\frac{41}{21}$	$\frac{41}{420}$	$\frac{41}{21}$
43	$\frac{43}{462}$	$\frac{43}{22}$	$\frac{43}{462}$	$\frac{43}{22}$
44	$\frac{1}{10}$	2	$\frac{1}{10}$	2
45	$\frac{1}{10}$	$\frac{25}{13}$	$\frac{1}{9}$	$\frac{3}{2}$
46	$\frac{1}{11}$	2	$\frac{1}{11}$	2

m	egalitarian ζ_{min}	egalitarian ζ_{max}	strongly egalitarian ζ_{min}	strongly egalitarian ζ_{max}
47	$\frac{47}{552}$	$\frac{47}{24}$	$\frac{47}{552}$	$\frac{47}{24}$
49	$\frac{1}{12}$	$\frac{49}{25}$	$\frac{1}{12}$	$\frac{49}{25}$
50	$\frac{1}{10}$	2	$\frac{1}{10}$	2
51	$\frac{3}{34}$	$\frac{126}{65}$	$\frac{1}{11}$	$\frac{3}{2}$
52	$\frac{1}{12}$	2	$\frac{1}{12}$	2
53	$\frac{53}{702}$	$\frac{53}{27}$	$\frac{53}{702}$	$\frac{53}{27}$
55	$\frac{5}{66}$	$\frac{218}{111}$	$\frac{1}{12}$	$\frac{49}{25}$
56	$\frac{1}{12}$	2	$\frac{1}{12}$	2
57	$\frac{3}{38}$	$\frac{134}{69}$	$\frac{1}{12}$	$\frac{3}{2}$
58	$\frac{1}{14}$	2	$\frac{1}{14}$	2
59	$\frac{59}{870}$	$\frac{59}{30}$	$\frac{59}{870}$	$\frac{59}{30}$
61	$\frac{61}{930}$	$\frac{61}{31}$	$\frac{61}{930}$	$\frac{61}{31}$
62	$\frac{1}{15}$	2	$\frac{1}{15}$	2
63	$\frac{1}{14}$	$\frac{113}{58}$	$\frac{1}{12}$	$\frac{3}{2}$
64	$\frac{1}{16}$	2	$\frac{1}{16}$	2
65	$\frac{5}{78}$	$\frac{65}{33}$	$\frac{1}{15}$	$\frac{65}{33}$
67	$\frac{67}{1122}$	$\frac{67}{34}$	$\frac{67}{1122}$	$\frac{67}{34}$
68	$\frac{1}{16}$	2	$\frac{1}{16}$	2
69	$\frac{3}{46}$	$\frac{45}{23}$	$\frac{1}{15}$	$\frac{3}{2}$
70	$\frac{1}{10}$	2	$\frac{1}{7}$	2
71	$\frac{71}{1260}$	$\frac{71}{36}$	$\frac{71}{1260}$	$\frac{71}{36}$
73	$\frac{73}{1332}$	$\frac{73}{37}$	$\frac{73}{1332}$	$\frac{73}{37}$
74	$\frac{1}{18}$	2	$\frac{1}{18}$	2
75	$\frac{3}{50}$	$\frac{166}{85}$	$\frac{1}{15}$	$\frac{3}{2}$
76	$\frac{1}{18}$	2	$\frac{1}{18}$	2
77	$\frac{7}{132}$	$\frac{77}{39}$	$\frac{1}{18}$	$\frac{77}{39}$
79	$\frac{79}{1560}$	$\frac{79}{40}$	$\frac{79}{1560}$	$\frac{79}{40}$
80	$\frac{1}{16}$	2	$\frac{1}{16}$	2
81	$\frac{1}{18}$	$\frac{63}{32}$	$\frac{1}{18}$	$\frac{3}{2}$
82	$\frac{1}{20}$	2	$\frac{1}{20}$	2
83	$\frac{83}{1722}$	$\frac{83}{42}$	$\frac{83}{1722}$	$\frac{83}{42}$
85	$\frac{5}{102}$	$\frac{85}{43}$	$\frac{1}{20}$	$\frac{85}{43}$
86	$\frac{1}{21}$	2	$\frac{1}{21}$	2
87	$\frac{3}{58}$	$\frac{234}{119}$	$\frac{1}{19}$	$\frac{3}{2}$
88	$\frac{1}{20}$	2	$\frac{1}{20}$	2
89	$\frac{89}{1980}$	$\frac{89}{45}$	$\frac{89}{1980}$	$\frac{89}{45}$

Table 1: Bounds for normalized egalitarian and strongly egalitarian weights 4...89

m	egalitarian		strongly egalitarian	
	ζ_{min}	ζ_{max}	ζ_{min}	ζ_{max}
91	$\frac{7}{156}$	$\frac{91}{46}$	$\frac{1}{21}$	$\frac{91}{46}$
92	$\frac{1}{22}$	2	$\frac{1}{22}$	2
93	$\frac{3}{62}$	$\frac{230}{117}$	$\frac{1}{20}$	$\frac{3}{2}$
94	$\frac{1}{23}$	2	$\frac{1}{23}$	2
95	$\frac{5}{114}$	$\frac{95}{48}$	$\frac{1}{22}$	$\frac{95}{48}$
97	$\frac{97}{2352}$	$\frac{97}{49}$	$\frac{97}{2352}$	$\frac{97}{49}$
98	$\frac{1}{21}$	2	$\frac{1}{21}$	2
99	$\frac{1}{22}$	$\frac{156}{79}$	$\frac{1}{21}$	$\frac{3}{2}$
100	$\frac{1}{20}$	2	$\frac{1}{20}$	2
101	$\frac{101}{2550}$	$\frac{101}{51}$	$\frac{101}{2550}$	$\frac{101}{51}$
103	$\frac{103}{2652}$	$\frac{103}{52}$	$\frac{103}{2652}$	$\frac{103}{52}$
104	$\frac{1}{24}$	2	$\frac{1}{24}$	2
105	$\frac{3}{70}$	$\frac{211}{108}$	$\frac{25}{482}$	$\frac{3}{2}$
106	$\frac{1}{26}$	2	$\frac{1}{26}$	2
107	$\frac{107}{2862}$	$\frac{107}{54}$	$\frac{107}{2862}$	$\frac{107}{54}$
109	$\frac{109}{2970}$	$\frac{109}{55}$	$\frac{109}{2970}$	$\frac{109}{55}$
110	$\frac{1}{22}$	2	$\frac{1}{20}$	2
111	$\frac{3}{74}$	$\frac{278}{141}$	$\frac{1}{24}$	$\frac{3}{2}$
112	$\frac{1}{24}$	2	$\frac{1}{24}$	2
113	$\frac{113}{3192}$	$\frac{113}{57}$	$\frac{113}{3192}$	$\frac{113}{57}$
115	$\frac{5}{138}$	$\frac{115}{58}$	$\frac{1}{27}$	$\frac{115}{58}$
116	$\frac{1}{28}$	2	$\frac{1}{28}$	2
117	$\frac{1}{26}$	$\frac{160}{81}$	$\frac{1}{24}$	$\frac{3}{2}$
118	$\frac{1}{29}$	2	$\frac{1}{29}$	2
119	$\frac{7}{204}$	$\frac{119}{60}$	$\frac{1}{28}$	$\frac{119}{60}$
121	$\frac{1}{30}$	$\frac{121}{61}$	$\frac{1}{30}$	$\frac{121}{61}$
122	$\frac{1}{30}$	2	$\frac{1}{30}$	2
123	$\frac{3}{82}$	$\frac{342}{173}$	$\frac{1}{27}$	$\frac{3}{2}$
124	$\frac{1}{30}$	2	$\frac{1}{30}$	2
125	$\frac{1}{30}$	$\frac{125}{63}$	$\frac{1}{30}$	$\frac{125}{63}$
127	$\frac{127}{4032}$	$\frac{127}{64}$	$\frac{127}{4032}$	$\frac{127}{64}$
128	$\frac{1}{32}$	2	$\frac{1}{32}$	2
129	$\frac{3}{86}$	$\frac{326}{165}$	$\frac{1}{28}$	$\frac{3}{2}$
130	$\frac{1}{26}$	2	$\frac{1}{24}$	2
131	$\frac{131}{4290}$	$\frac{131}{66}$	$\frac{131}{4290}$	$\frac{131}{66}$
133	$\frac{7}{228}$	$\frac{133}{67}$	$\frac{1}{32}$	$\frac{133}{67}$

m	egalitarian		strongly egalitarian	
	ζ_{min}	ζ_{max}	ζ_{min}	ζ_{max}
134	$\frac{1}{33}$	2	$\frac{1}{33}$	2
135	$\frac{1}{30}$	$\frac{608}{307}$	$\frac{1}{27}$	$\frac{3}{2}$
136	$\frac{1}{32}$	2	$\frac{1}{32}$	2
137	$\frac{137}{4692}$	$\frac{137}{69}$	$\frac{137}{4692}$	$\frac{137}{69}$
139	$\frac{139}{4830}$	$\frac{139}{70}$	$\frac{139}{4830}$	$\frac{139}{70}$
140	$\frac{1}{20}$	2	$\frac{1}{14}$	2
141	$\frac{3}{94}$	$\frac{99}{50}$	$\frac{1}{31}$	$\frac{3}{2}$
142	$\frac{1}{35}$	2	$\frac{1}{35}$	2
143	$\frac{11}{390}$	$\frac{143}{72}$	$\frac{1}{35}$	$\frac{143}{72}$
145	$\frac{5}{174}$	$\frac{145}{73}$	$\frac{1}{34}$	$\frac{145}{73}$
146	$\frac{1}{36}$	2	$\frac{1}{36}$	2
147	$\frac{3}{98}$	$\frac{374}{189}$	$\frac{1}{28}$	$\frac{3}{2}$
148	$\frac{1}{36}$	2	$\frac{1}{36}$	2
149	$\frac{149}{5550}$	$\frac{149}{75}$	$\frac{149}{5550}$	$\frac{149}{75}$
151	$\frac{151}{5700}$	$\frac{151}{76}$	$\frac{151}{5700}$	$\frac{151}{76}$
152	$\frac{1}{36}$	2	$\frac{1}{36}$	2
153	$\frac{1}{34}$	$\frac{123}{62}$	$\frac{1}{33}$	$\frac{3}{2}$
154	$\frac{1}{33}$	2	$\frac{4}{129}$	2
155	$\frac{5}{186}$	$\frac{155}{78}$	$\frac{1}{36}$	$\frac{155}{78}$
157	$\frac{157}{6162}$	$\frac{157}{79}$	$\frac{157}{6162}$	$\frac{157}{79}$
158	$\frac{1}{39}$	2	$\frac{1}{39}$	2
159	$\frac{3}{106}$	$\frac{450}{227}$	$\frac{1}{35}$	$\frac{3}{2}$
160	$\frac{1}{32}$	2	$\frac{1}{32}$	2
161	$\frac{7}{276}$	$\frac{161}{81}$	$\frac{1}{39}$	$\frac{161}{81}$
163	$\frac{163}{6642}$	$\frac{163}{82}$	$\frac{163}{6642}$	$\frac{163}{82}$
164	$\frac{1}{40}$	2	$\frac{1}{40}$	2
165	$\frac{3}{110}$	$\frac{378}{191}$	$\frac{1}{30}$	$\frac{3}{2}$
166	$\frac{1}{41}$	2	$\frac{1}{41}$	$\frac{81}{41}$
167	$\frac{167}{6972}$	$\frac{167}{84}$	$\frac{167}{6972}$	$\frac{167}{84}$
169	$\frac{1}{42}$	$\frac{169}{85}$	$\frac{1}{42}$	$\frac{169}{85}$
170	$\frac{1}{34}$	$\frac{67}{34}$	$\frac{1}{32}$	$\frac{63}{32}$
171	$\frac{1}{38}$	$\frac{119}{60}$	$\frac{1}{36}$	$\frac{3}{2}$
172	$\frac{1}{42}$	2	$\frac{1}{42}$	2
173	$\frac{173}{7482}$	$\frac{173}{87}$	$\frac{173}{7482}$	$\frac{173}{87}$
175	$\frac{1}{42}$	$\frac{163}{82}$	$\frac{1}{40}$	$\frac{139}{70}$
176	$\frac{1}{40}$	$\frac{79}{40}$	$\frac{1}{40}$	2

Table 2: Bounds for normalized egalitarian and strongly egalitarian weights 91...176

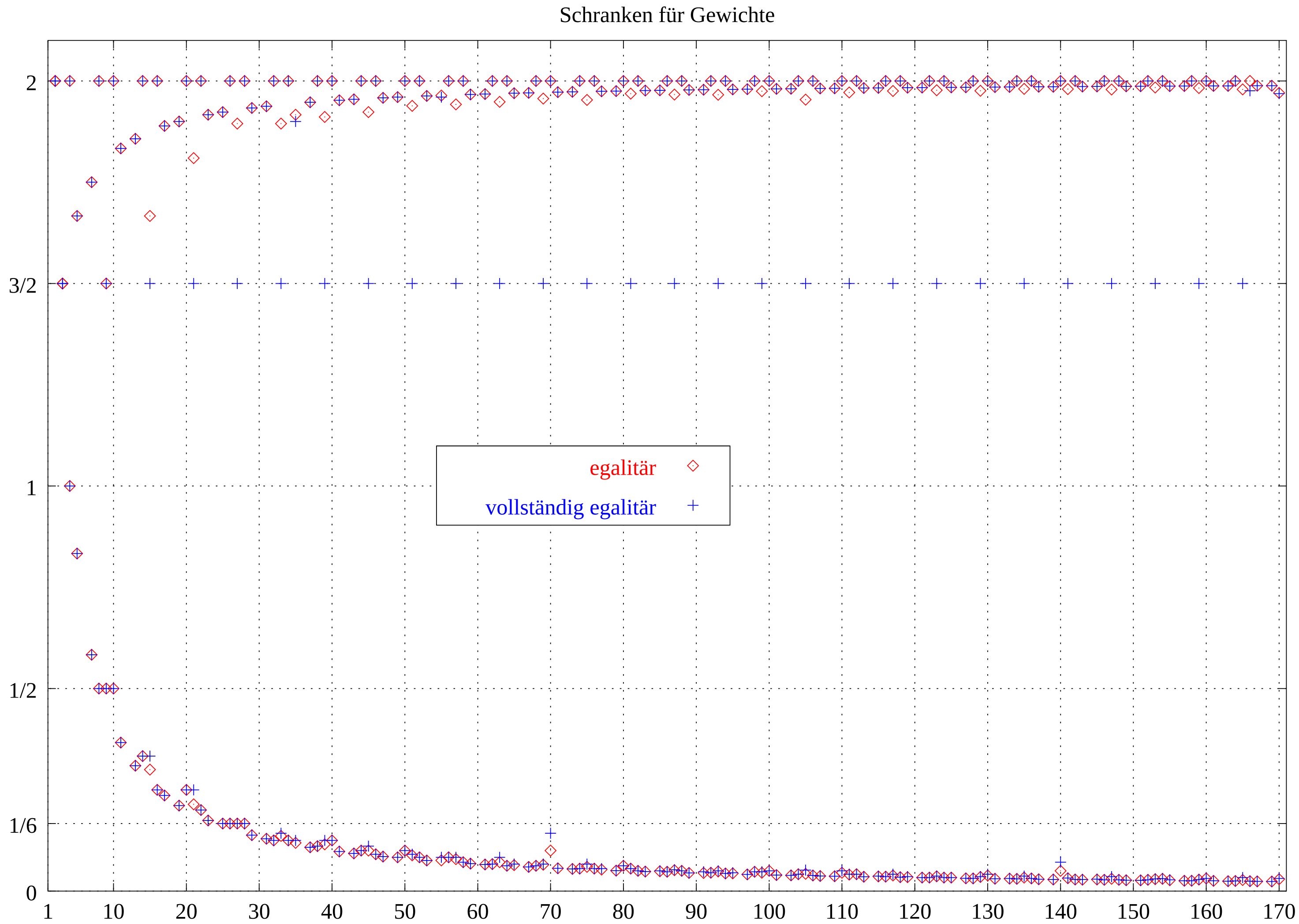


Figure 12: Bounds for normalized egalitarian and strongly egalitarian weights 2 . . . 170

	${}_{min}\zeta_{egalitarian}$ min. of egal. weight	${}^{max}\zeta_{egalitarian}$ max. of egal. weight	${}_{min}\zeta^{strongly}_{egalitarian}$ min. of str. egal. weight	${}^{max}\zeta^{strongly}_{egalitarian}$ max. of str. egal. weight
$m = 3 \cdot k;\ k \in \mathbb{N}$				${}^{max}\zeta^{strongly}_{egalitarian} = \frac{3}{2}$
odd $m \in \mathbb{P}$	egalitarian is the same as strongly egalitarian $x_{min} = \begin{pmatrix} x_1 = \frac{4m}{m^2-1} \\ x_2 = 2x_1 \\ x_3 = x_1 + x_2 \\ \vdots \\ x_k = x_1 + x_{k-1} \\ \vdots \\ x_{\lfloor \frac{m}{2} \rfloor} = x_1 + x_{\lfloor \frac{m}{2} \rfloor - 1} \end{pmatrix};\ x_{max} = \begin{pmatrix} x_1 = \frac{2m}{m+1} \\ x_2 = \frac{m}{m+1} \\ \vdots \\ x_{\lfloor \frac{m}{2} \rfloor} = \frac{m}{m+1} \end{pmatrix}$			
$m = 2^k;\ k \in \mathbb{N}$	egalitarian is the same as strongly egalitarian ${}_{min}\zeta_{egalitarian} = {}_{min}\zeta^{strongly}_{egalitarian} = \begin{cases} \frac{4}{m} & \text{with } m \equiv 0 \pmod 2 \\ \frac{4m}{m^2-1} & \text{with } m \equiv 1 \pmod 2 \end{cases}$ ${}^{max}\zeta_{egalitarian} = {}^{max}\zeta^{strongly}_{egalitarian} = 2 = w(2^{k-1})$ example: Lee weight $w_{Lee} : \mathbb{Z}_m \to \mathbb{Q},\ x \mapsto \min\{x, m-x\}$ with $\zeta(w_{Lee}) = \frac{m^2-1}{4m}$			
$m = p^k;\ p \in \mathbb{P};\ k \in \mathbb{N}$			${}_{min}\zeta_{egalitarian} = \frac{4}{p^{k-2}(p^2-1)}$ example: Vikhren weight $w_{Vikh}(x) = \begin{cases} w_{Lee}(x) & \text{with } x \leq \frac{p-1}{2} \cdot p^{k-1} \vee x \geq \frac{p+1}{2} \cdot p^{k-1} \\ \frac{p-1}{2} \cdot p^{k-1} & \text{with } \frac{p-1}{2} \cdot p^{k-1} < x < \frac{p+1}{2} \cdot p^{k-1} \end{cases}$	

Table 3: General facts for the bounds

- $\forall m : {}_{min}\zeta_{egalitarian}(m) = \dfrac{p}{m} \cdot {}_{min}\zeta_{egalitarian}(p)$ with $p = \min\{x \in \mathbb{P} : x|m\} \geq 5$
- $\forall m : {}_{min}\zeta_{egalitarian}(m) = \dfrac{\overline{m}}{m} \cdot {}_{min}\zeta_{egalitarian}(\overline{m})$

  $\forall m : {}_{min}\zeta^{strongly}_{egalitarian}(m) = \dfrac{\overline{m}}{m} \cdot {}_{min}\zeta^{strongly}_{egalitarian}(\overline{m})$ with $\overline{m} = p_1 \cdot p_2 \cdot \ldots \cdot p_n$ the quadratic free core of $m = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_n^{k_n}$
- $\forall m = 2^j \cdot p^k$ with $p \in \mathbb{P}$, $p > 3$ and $j, k \in \mathbb{N} : {}_{min}\zeta_{egalitarian} = {}_{min}\zeta^{strongly}_{egalitarian} = \dfrac{2}{\phi(m)} = \dfrac{2^{2-j}}{p^{k-1} \cdot (p-1)} = \dfrac{4 \cdot p}{m \cdot (p-1)}$
- $\forall m = 3 \cdot p$ with $p \in \mathbb{P}$ and $p > 3 : {}_{min}\zeta_{egalitarian} = \dfrac{3}{\phi(m) + 2} = \dfrac{3}{2 \cdot p}$
- $\forall m = p_1 \cdot p_2$ with $p_1, p_2 \in \mathbb{P}$ and $3 \leq p_1 \leq p_2 : {}_{min}\zeta_{egalitarian}(m) = \dfrac{1}{p_2} \cdot {}_{min}\zeta_{egalitarian}(p_1)$

Figure 13: Hypothesis for the bounds

5 Simplifications

Theorem 5.1 (weight vector) *To determine a weight vector $x \in \mathbb{R}^m$ it is sufficient to determine x_i for $i \in I := [1; \frac{m}{2}] \cap \mathbb{Z}$ with the help of:*

(G1) $\forall i \in I : x_i > 0$

(G2) $\forall i, j \in I$ *with* $i \leq j \wedge j \neq m - i \wedge j \neq m - 2i \wedge j \neq \frac{m-i}{2}$:

$$\begin{array}{l} \textit{for } i + j \leq \left\lfloor \frac{m}{2} \right\rfloor : x_{i+j} \leq x_i + x_j \\ \textit{for } i + j > \left\lfloor \frac{m}{2} \right\rfloor : x_{m-(i+j)} \leq x_i + x_j \end{array}$$

(G3) $\forall i, j \in I$ *with* $i < j \wedge j \neq 2i \wedge j \neq \frac{m}{2}$:

$$x_{j-i} \leq x_i + x_j$$

The left over components are given by:

$$\forall i \notin I : x_i = x_{m-i} \textit{ where is } m - i \in I$$

Theorem 5.2 (egalitärer Gewichtsvektor) *To determine a normalized egalitarian weight vector $x \in \mathbb{R}^m$ it is sufficient to determine x_i for $i \in I := [1; \frac{m}{2}] \cap \mathbb{Z}$ with the help of:*

(E1) *x fulfill (G1),(G2) and (G3) from theorem 5.1*

(E2) *the following equations have to be fulfilled:*
for m even:

$$x_{\frac{m}{2}} = 2$$

$$\forall d|m \text{ with } d \geq 3 > 2:$$

$$\sum_{i=2}^{\lfloor \frac{d}{2} \rfloor + 1} x_{(i-1)\frac{m}{d}} + \sum_{i=\lfloor \frac{d}{2} \rfloor + 2}^{d} x_{m(1+\frac{1-i}{d})} = d$$

for m odd:

$$\forall d|m \text{ with } d \geq 3:$$

$$\sum_{i=2}^{\lfloor d\frac{m-1}{2m} \rfloor + 1} x_{(i-1)\frac{m}{d}} + \sum_{i=\lfloor d\frac{m-1}{2m} \rfloor + 2}^{d} x_{m(1+\frac{1-i}{d})} = d$$

The left over components are given by:

$$\forall i \notin I : x_i = x_{m-i} \text{ where is } m - i \in I$$

Theorem 5.3 (vollständig egalitärer Gewichtsvektor) *To determine a normalized strongly egalitarian weight vector $x \in \mathbb{R}^m$ it is sufficient to determine x_i for $i \in I := [1; \frac{m}{2}] \cap \mathbb{Z}$ with the help of:*

(VE1) *x fulfill (G1),(G2) and (G3) from theorem 5.1*

(VE2) *$\forall d|m$ with $d \in \mathbb{P}$:* [13] [14]

$$\sum_{i=2}^{\lfloor \frac{d}{2} \rfloor + 1} x_{(i-1)\frac{m}{d}} + \sum_{i=\lfloor \frac{d}{2} \rfloor + 2}^{d} x_{m-(i-1)\frac{m}{d}} = d$$

$$\forall z \in \left[1; \frac{m}{2d}\right] \cap \mathbb{Z} : \sum_{i=1}^{\lfloor \frac{d}{2} - \frac{z\,d}{m} \rfloor + 1} x_{z+(i-1)\frac{m}{d}} + \sum_{i=\lfloor \frac{d}{2} - \frac{z\,d}{m} \rfloor + 2}^{d} x_{m-(z+(i-1)\frac{m}{d})} = d$$

The left over components are given by:

$\forall i \notin I : x_i = x_{m-i}$ *where is* $m - i \in I$

□

[13] The number 1 isn't a prime number: $1 \notin \mathbb{P}$

[14] The set $[1; \frac{m}{2d}]$ is empty for $\frac{m}{2d} < 1$ by appointment.

A Literature, List of Figures

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